290 - 1295 To Bob Collins

2009 "Wayne" Bowl

(This year's Math Bowl event held in honor of Dr. Wayne Roberts)

1. Three primes, p, q, and r satisfy $p+q=r, \quad 1<p<q$. What is p?

(30 seconds)

SOLUTION

p and q cannot both be odd; therefore, $\boxed{p=2}$, $q=3$, and $r=5$.

The number of years, out of the 29 years of existence of the Minnesota State High School Mathematics League, in which Wayne Roberts has <u>not</u> been the primary problem writer for the League. (Loren Larson, of St. Olaf, wrote the problems for 2 years while Wayne was serving as Provost of Macalester College.)

---

2. The first four terms of an arithmetic sequence are a, $2x$, b, $5x$. What is the ratio b : a?

(45 seconds)

SOLUTION

Let the common difference be d.

$2x+2d=5x \Rightarrow d=\dfrac{3x}{2}$. So $a=2x-\dfrac{3x}{2}=\dfrac{x}{2}$, and $b=2x+\dfrac{3x}{2}=\dfrac{7x}{2}$.

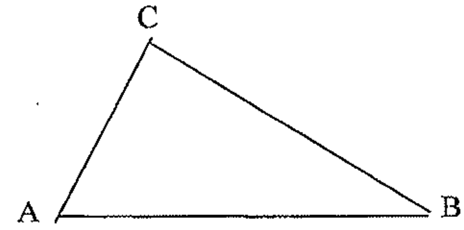
Thus $\dfrac{b}{a}=\dfrac{7x/2}{x/2}=\boxed{7}$.

Minnesota's AHSME region number in 1973, ranking 10th out of the 10 U.S. regions. At this time, Wayne took over as Minnesota's Coordinator for the AHSME, studying results which would later lead to the formation of the MSHSML.

On the 2008 AMC 12 exam, Minnesota had 6 of the top 12 scores in its new region, Region 5, which covers 9 states (MN, WI, IA, ND, SD, KS, NE, OK, AR).

3. In the right ΔABC, $AC = \frac{8\sqrt{3}}{3}$ and $BC = 8$.

What is the length of the altitude dropped from C?



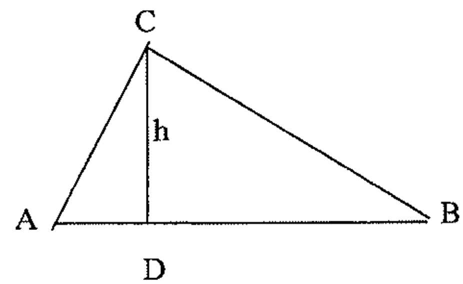
(60 seconds)

SOLUTION

First notice that ΔABC is 30-60-90, so ΔACD must be also.

Let the altitude be $h = CD$.

Then $h = \frac{AC}{2} \cdot \sqrt{3} = \frac{4\sqrt{3}}{3} \cdot \sqrt{3} = \boxed{4}$.



The number of founding schools in 1980-81, our League's inaugural season. The schools were: Minneapolis Southwest, Minneapolis Edison, St. Paul Como, and St. Paul Central.

---

4. A parabola $y = ax^2 + bx + c$ has vertex (4, 2). If (2, 0) is on the parabola, find the smallest possible integral product *abc*.

(90 seconds)

SOLUTION

This parabola has zeroes of 2 and 6. Therefore, $y = a(x-2)(x-6)$.

Substitute $x = 4$, $y = 2$ to determine $a = -\frac{1}{2}$.

$$y = -\frac{1}{2}(x-2)(x-6) = -\frac{1}{2}\left(x^2 - 8x + 12\right) = -\frac{1}{2}x^2 + 4x - 6$$

The smallest possible product is $\left(-\frac{1}{2}\right)(4)(-6) = \boxed{12}$.

The number of schools added to the League in its second season, 1981-82. Within five years, the number of teams exceeded 100. There are currently 173 MSHSML teams!

5. Find the measure, in degrees, of the smallest positive angle x for which $\sin 4x \sin 11x = \cos 11x \cos 4x$.

(90 seconds)

SOLUTION

$\sin 4x \sin 11x = \cos 11x \cos 4x \Rightarrow \cos 4x \cos 11x - \sin 4x \sin 11x = 0$.

Since $\cos 4x \cos 11x - \sin 4x \sin 11x = \cos(4x + 11x)$, we have

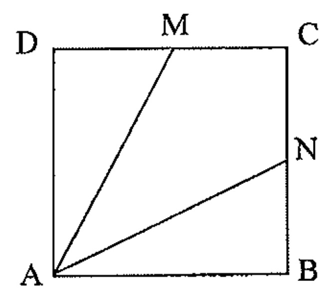
$\cos 15x = 0 \Rightarrow 15x = 90° \Rightarrow x = \boxed{6}$ degrees.

The number of letters in A. Wayne Roberts's <u>actual</u> first name (it's Arthur) !

---

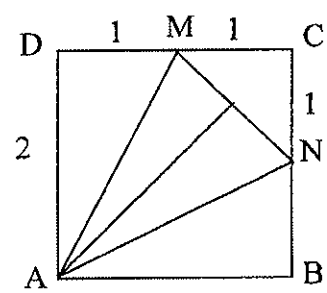
6. The figure at the right shows square $ABCD$. M is the midpoint of CD; N is the midpoint of BC. Find $\sin(\angle MAN)$.

(90 seconds)



SOLUTION

Without loss of generality, suppose the sides of the square have length 2, so that $MN = \sqrt{2}$ and $AM = \sqrt{5}$. Then:



$$\sin\left(\frac{\angle MAN}{2}\right) = \frac{\sqrt{2}/2}{\sqrt{5}} = \frac{1}{\sqrt{10}}$$

$$\sin(\angle MAN) = 2\sin\left(\frac{\angle MAN}{2}\right)\cos\left(\frac{\angle MAN}{2}\right) = 2 \cdot \frac{1}{\sqrt{10}} \cdot \frac{3}{\sqrt{10}} = \boxed{\frac{3}{5}}.$$

The fraction of regular season meets each year in which Wayne writes a problem based on the League logo at the top of the exams.

7. There are $5! = 120$ "words" that can be formed using the letters A, E, N, W, Y. Suppose these 120 words were listed in alphabetical order (i.e., the first three words would be AENWY, AENYW, AEWNY).
What number in the list would the word WAYNE be?

(90 seconds)

SOLUTION

Words beginning with A	$4! = 24$
Words beginning with E	$4! = 24$
Words beginning with N	$4! = 24$

Now we are to the WA __ __ __ words. Begin listing them:
WAENY, WAEYN, WANEY, WANYE, WAYEN, **WAYNE**

"WAYNE" is in position $24 + 24 + 24 + 6 = \boxed{78}$.

The year (1978) in which Wayne Roberts first learned about the existence of student math leagues, while living in Massachusetts.

---

8. Each regular-season MSHSML meet consists of four 4-question event and one 6-question team event. Each MSHSML State Tournament meet is like a regular-season meet, but with an additional 14-question Invitational event.

In the 29 years of the MSHSML, Wayne Roberts has written the problems for 27 five-meet regular seasons (minus a total of six meets over the past two years in which the new members of the writing team wrote 75% of the problems). He has also written 23 years worth of State Tournament meets, 14 of which also included the 15-question Math Bowl.

Calculate the total number of problems Wayne Roberts has written for our League. (...and don't forget to subtract 1, since this problem was written <u>for him</u>!)

(2 minutes)

SOLUTION

$$27\ \cancel{seasons} \cdot \frac{5\ \cancel{meets}}{\cancel{season}} \cdot \frac{22\ questions}{\cancel{meet}} = 2970\ questions$$

$$6\ \cancel{meets} \cdot \frac{22\ questions}{\cancel{meet}} \cdot .75 = 99\ questions \text{ written \underline{not} by Wayne}$$

$$23\ \cancel{Tournaments} \cdot \frac{36\ questions}{\cancel{Tournament}} = 828\ questions$$

$$14\ \cancel{MathBowls} \cdot \frac{15\ questions}{\cancel{MathBowl}} = 210\ questions$$

$$2970 - 99 + 828 + 210 - 1 = \boxed{3908} \text{ problems.}$$

That number speaks for itself.

---

9. Find the sum of all real roots of $x^{\log x} = 10$.

(60 seconds)

SOLUTION

$$x^{\log x} = 10$$
$$\log x^{\log x} = 1$$
$$[\log x]^2 = 1$$
$$\log x = 1 \text{ or } \log x = -1$$
$$x = 10 \text{ or } x = 0.1$$

$$\text{Sum} = 10 + 0.1 = \boxed{10.1}$$

10. A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks, and 40 black socks. A youngster selects socks one at a time from the drawer, but is unable to see the color of the socks drawn. What is the smallest number of socks that must be selected to guarantee that the selection contains 10 pairs? (A pair is two socks of the same color. No sock may be counted in more than one pair.)

(60 seconds)

SOLUTION

After each draw, one has a maximum of 4 unmatched socks.
Suppose one had 23 socks. Can 4 be unmatched, 19 matched?
Of course not. We cannot have 19 that are matched.
If four are unmatched, we could have 18 (9 pair) matched.
That would be 22 socks in all.
The next draw, the 23rd, will produce the 10th pair.

$\boxed{23}$ socks must be drawn.

---

11. $p(x)=x^2+bx+c$ is a factor of both x^4+6x^2+25 and $3x^4+4x^2+28x+5$. Find $p(1)$.

(2 minutes)

SOLUTION

$$\begin{aligned}
p(x)s(x) &= 3x^4+4x^2+28x+5\\
p(x)r(x) &= x^4+6x^2+25\\
3\cdot p(x)r(x) &= 3x^4+18x^2+75\\
p(x)s(x)-3\cdot p(x)r(x) &= -14x^2+28x-70\\
p(x)\left[s(x)-3\cdot r(x)\right] &= -14\left(x^2-2x+5\right)\\
\therefore\ p(x) &= x^2-2x+5\ ;
\end{aligned}$$

$p(1)=\boxed{4}$.

12. Six distinct integers are picked at random from the set $\{1, 2, 3, \ldots .10\}$.

What is the probability that 3 is the second smallest of the integers selected?

(90 seconds)

SOLUTION

There are $\binom{10}{6} = 210$ ways to choose the six integers.

Either 1 or 2 must be the smallest integer chosen; there are just 2 ways to make this choice.

Of course 3 must be chosen.

There are $\binom{7}{4} = 35$ ways to choose four more integers from $\{4, 5, ..10\}$.

Thus, there are (2)(35) = 70 ways to be successful.

The probability of success is $\frac{70}{210} = \boxed{\frac{1}{3}}$.

---

13. Find the product of all x that satisfy $\left|x - |2x+1|\right| = 3$.

(90 seconds)

SOLUTION

If $2x+1 \geq 0$ so $x \geq -\frac{1}{2}$, then $|x-(2x+1)| = |x+1| = 3$,

and $x = 2$ $\left(x = -4 \text{ does not satisfy } x \geq -\frac{1}{2}\right)$.

If $2x+1 \leq 0$ so $x \leq -\frac{1}{2}$, then $|x+(2x+1)| = |3x+1| = 3$

So $x = -\frac{4}{3}$ $\left(x = \frac{2}{3} \text{ does not satisfy } x \leq -\frac{1}{2}\right)$.

The product is $(2)\left(\frac{-4}{3}\right) = \boxed{\frac{-8}{3}}$.

14. A sequence $\{a_n\}$ is defined by $a_1 = 2; \quad a_{n+1} = a_n + 2n$. Find a_{100}.

(2 minutes)

SOLUTION

$$
\begin{aligned}
a_2 - a_1 &= 2\cdot 1 \\
a_3 - a_2 &= 2\cdot 2 \\
a_3 - a_2 &= 2\cdot 2 \\
\vdots \quad \vdots &= \vdots \; \vdots \\
a_{100} - a_{99} &= 2\cdot 99
\end{aligned}
$$

Adding,

$$a_{100} - a_1 = 2(1 + \ldots + 99) = 2\frac{(100)(99)}{2} \qquad a_{100} = \boxed{9902}.$$

$$a_{100} - 2 = 9900$$

---

15. What is the value of $\sqrt{3+\sqrt{3+\sqrt{3+\ldots}}}$?

(first correct answer)

SOLUTION

Let $x = \sqrt{3+\sqrt{3+\sqrt{3+\ldots}}}$ Then $x+3 = 3+\sqrt{3+\sqrt{3+\sqrt{3+\ldots}}}$

Take the square root of both sides: $\sqrt{x+3} = \sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\ldots}}}} = x$

So $x+3 = x^2 \Rightarrow x^2 - x - 3 = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{1+12}}{2}$, and eliminating the extraneous root, $x = \boxed{\dfrac{1+\sqrt{13}}{2}}$.